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Irrelevant interactions without composite operators: a remark on the universality of second-order phase transitions

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Abstract

We study the critical behaviour of symmetric ϕ_4^4 theory including irrelevant terms of the form $\phi^{4+2n}/\Lambda_0^{2n}$ in the bare action, where Λ_0 is the UV cutoff (corresponding, e.g., to the inverse lattice spacing for a spin system). The main technical tool is renormalization theory based on the flow equations of the renormalization group, which permits us to establish the required convergence statements in generality and rigour. As a consequence the effect of irrelevant terms on the critical behaviour may be studied to any order without using renormalization theory for composite operators. This is a technical simplification and seems preferable from the physical point of view. In this short paper we restrict ourselves for simplicity to the symmetry class of the Ising model, i.e. one-component ϕ_4^4 theory. The method is general, however.

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1. Introduction

One of the great achievements of theoretical physics in the 1970s was the unification of concepts and ideas from quantum field theory and statistical mechanics through the Wilson renormalization group [WiKo]. In particular renormalized perturbation theory was applied successfully to the study of second-order phase transitions and to the calculation of critical exponents [BGZ, Amit, ZJ]. Somewhat later Polchinski [Pol] realized that renormalized perturbation theory itself could be based directly on the flow equations (FEs) of the renormalization group (an idea which had been proposed by Wilson himself in the late 1960s, as we learned from Brézin). Thus a physical problem, the control of the flow of effective actions under certain boundary conditions, replaced the analysis of Feynman diagrams with rather technical tools such as Zimmermann forests and Hepp sectors [Kop, Sal]. One of the challenging conceptual problems in the analysis of critical phenomena was the question of universality, i.e. to realize why large classes of theories, specified essentially by the respective

¹ Stagiaire.

Hamiltonians, should give rise to the same critical behaviour characterized through the critical exponents. Experimentally these depend only on dimensionality and symmetry and not on details of the dynamics. Modifications of the Hamiltonians thus should lead only to subleading corrections. We restrict our explicit presentation to one of the simplest and best known classes, that of the Ising model. The method is general, however. Passing to the continuous description, which should be viable for correlation lengths ξ much larger than the lattice spacing, i.e. in the vicinity of the critical point, the symmetry class of the Ising model is presented by ϕ^4 theory, symmetric under $\phi \rightarrow -\phi$. The standard action at the scale of the UV cutoff Λ_0 , corresponding to the inverse lattice spacing in position space, is then

$$L^0 = \int (a \phi^2(x) + b(\partial_\mu \phi)^2(x) + c \phi^4(x)) d^4x. \quad (1)$$

If we restrict ourselves to perturbation theory the constants a , b , c are to be viewed as power series in the renormalized coupling g or in \hbar . In the standard notation this expression is rewritten as

$$L^0 = \int \left(\frac{Z}{2} (\partial_\mu \phi)^2(x) + \frac{Z}{2} (m^2 + \delta m^2) \phi^2(x) + \frac{g_0}{4!} Z^2 \phi^4(x) \right) d^4x \quad (2)$$

where L^0 also includes the term of order zero in perturbation theory, that is to say

$$L^0 = L^0 + \int \left(\frac{1}{2} (\partial_\mu \phi)^2(x) + \frac{1}{2} m^2 \phi^2(x) \right) d^4x. \quad (3)$$

The field ϕ corresponds to the renormalized field. In (2) we introduced the standard notation for the wavefunction renormalization Z and the mass counterterm δm as well as the bare coupling g_0 . We restrict ourselves to the four-dimensional theory, which also serves to study lower-dimensional theories through the ε -expansion². Starting from the Ising model Hamiltonian on a cubic lattice one arrives at the action (1) on performing block spin transformations, expanding in local terms and passing to the continuous limit, on neglecting all irrelevant terms, i.e. those of mass dimension larger than four. The aim of this paper is to show that this is indeed justified when analysing long-distance phenomena near the critical point. This means that the dominant contributions to the correlation functions near the critical point are obtained from (1) for suitable choices of a , b , c . More precisely we will add a finite sum

$$A(\phi) = \int \sum_{n=1}^N \frac{Z^{2+n}}{\Lambda_0^{2n}} \frac{g_{4+2n}}{(4+2n)!} \phi^{4+2n}(x) d^4x \quad (4)$$

to (1). Here the UV cutoff Λ_0 appears naturally when expanding in local terms, by dimensional analysis. This means that the couplings g_{4+2n} are dimensionless (in $d = 4$)³. Since the statements of renormalization theory are generally of perturbative nature, i.e. valid on formal expansion in the couplings g , they require small values of these to be reliable. When including (4) the question arises of how the size of the irrelevant couplings compares to that of the original ϕ^4 coupling g . Here of course different situations may arise and can be analysed. Later on we will regard the situation where they are chosen such that the loop expansion remains valid, which means generally that

$$g_{4+2n} \sim g^{n+1}. \quad (5)$$

² Dimensional regularization cannot be naturally accommodated in the FE framework. Still the associated minimal renormalization schemes should be implementable. This has been shown for analytic regularization [KoSm].

³ We choose conventions such that a factor of Z^{2+n} appears in front of g_{4+2n} , which will somewhat simplify the notation later. Note that Z will depend on the couplings g_{4+2n} . In particular it will also be different from 1 for $g = 0$, if some of the g_{4+2n} do not vanish.

The expansion with respect to local terms also produces higher-dimensional terms of the form $(\phi^n \partial^w \phi^m)(x)$, which contain $|w|$ derivatives with respect to the coordinates x . Starting from a cubic lattice only terms respecting rotational symmetry, i.e. invariant under the Euclidean group, should appear in the continuum limit, i.e. when approaching the critical region. Furthermore in (4) only terms invariant under $\phi \rightarrow -\phi$ are generated if \mathbb{Z}_2 -symmetry is unbroken. For brevity of notation we restrict ourselves to (4); inclusion of derivative terms would only lead to minor changes. In the explicit treatment we will even limit ourselves to a single insertion $\sim g_6 \phi^6$, for simplicity of notation.

The effects of irrelevant terms have of course been studied extensively in the literature [Weg, BGZ] and can be found in textbooks, for example [ZJ, Chapter 26]. In the field theory approach these terms were analysed by renormalization theory for composite operators, as it existed in the early 1970s [Zim]. Treating ϕ^6 , for example, as a composite operator insertion means that one is restricted to Green functions carrying at most one insertion of this ϕ^6 term⁴. Then one has to fix renormalization conditions for the inserted Green functions up to dimension six (thus on the two-, four- and six-point functions and on derivatives of the two- and four-point functions). The general and probably optimal bound on the coefficient of the term ϕ^6 in L^0 is then of the form $\mathcal{P}(\log \Lambda_0)$, i.e. a polynomial in logarithms of Λ_0 —as has been shown, e.g., in [KeKo1]—and not $\sim \Lambda_0^{-2} \mathcal{P}(\log \Lambda_0)$ as in (4). Otherwise stated this means that in general it is not possible to find out the renormalization conditions which would give a bound $\sim \Lambda_0^{-2}$, since the associated dynamical system is unstable. From the physical point of view it seems therefore preferable to start directly from the modification of the *bare* action as in (4), and to perform the renormalization for this theory. We note however that it is not really possible to study the question in such a way that the *only* change in the bare action consists in adding the term $\sim \phi^6$ to it. The counterterms $\sim \phi^4$ and $\sim \phi^2$ change at the same time if we keep the renormalization conditions fixed. This phenomenon corresponds to what is called operator mixing in the theory of composite operators. However, whereas these renormalization conditions generally are related to the physical parametrization of the theory near the critical region and thus accessible to experiment, this seems not to be the case for the Green functions carrying, for example, ϕ^6 -insertions. So our novel results consist in presenting a framework which permits us to study *an arbitrary number of irrelevant insertions for a fixed set of renormalization conditions on the relevant parameters only*. Thus the study of these insertions is generalized and simplified at the same time as being compared to composite operator theory. We should also mention that, in a more phenomenological context, the FE framework has been widely used in recent years for calculations of critical exponents in various approximation schemes (see e.g. [Alf, MT, TW]; for a recent review see [BB]).

Renormalizability proofs based on the renormalization group are conceptually simple and rigorous and give a transparent view of the universality of critical behaviour, in showing that the modification of the action by irrelevant terms does not influence (the dominant part of) the critical behaviour. In this note we would like to make this explicit for the simplest case, the universality class of the Ising model. In the next section we will present the required results on the renormalizability of the theories with and without ϕ^6 -insertion, in particular in the critical region. In the last section we also use the (standard) renormalization group equations to analyse the subdominant contributions of the irrelevant insertion to the critical behaviour.

⁴ It is possible to go beyond one insertion, but then the number of renormalization conditions one has to fix increases with the number of insertions, corresponding to the fact that a ϕ^6 theory is nonrenormalizable.

2. Renormalization of ϕ_4^4 -theory with irrelevant terms

Renormalization theory as we are going to use it here is based on the FEs of the renormalization group due to Wilson [WiKo], and particularly to Polchinski [Pol] as regards the application to the perturbative renormalization problem. The FE is obtained by successively integrating out momenta in the (regularized) theory starting from the UV cutoff Λ_0 down to the scale $\Lambda < \Lambda_0$. The final renormalized theory is obtained on taking the limits $\Lambda_0 \rightarrow \infty$ and $\Lambda \rightarrow 0$. Its differential form can be obtained when deriving with respect to Λ the generating functional of the connected (free propagator) amputated Green (CAG) functions of the theory with momenta restricted to lie between Λ and Λ_0 . The scales Λ and Λ_0 enter through the regularized propagator, which for the massive theory takes the form

$$C^{\Lambda, \Lambda_0}(p) = \frac{1}{p^2 + m^2} \left\{ \exp\left(-\frac{p^2 + m^2}{\Lambda_0^2}\right) - \exp\left(-\frac{p^2 + m^2}{\Lambda^2}\right) \right\}. \quad (6)$$

This expression seems unnecessarily complicated for a UV regularization. The additional parameter Λ is however indispensable for the study of the renormalization group flow, and it is a physical parameter in this sense.

Its Fourier transform is

$$\hat{C}^{\Lambda, \Lambda_0}(x) = \int_p C^{\Lambda, \Lambda_0}(p) e^{ipx}. \quad (7)$$

We use the conventions⁵ $\int_p := \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4}$, $\phi(x) = \int_p \varphi(p) e^{ipx}$, $\frac{\delta}{\delta\phi(x)} = (2\pi)^4 \int_p \frac{\delta}{\delta\varphi(p)} e^{-ipx}$.

For finite Λ_0 and in finite volume the theory can be given rigorous meaning starting from the functional integral

$$\exp\left(-\frac{1}{\hbar}(L^{\Lambda, \Lambda_0}(\phi) + I^{\Lambda, \Lambda_0})\right) = \int d\mu_{\Lambda, \Lambda_0}(\Phi) \exp\left(-\frac{1}{\hbar}L^{\Lambda_0, \Lambda_0}(\phi + \Phi)\right) \quad (8)$$

where the factors of \hbar have been introduced to allow for a consistent loop expansion in the following, which permits us to stay with a single expansion parameter in the presence of two coupling constants. In (8) $d\mu_{\Lambda, \Lambda_0}(\Phi)$ denotes the (translation invariant) Gaussian measure with covariance $\hbar\hat{C}^{\Lambda, \Lambda_0}(x)$. The normalization factor $e^{-\frac{1}{\hbar}I^{\Lambda, \Lambda_0}}$ is due to vacuum contributions. It diverges in infinite volume so that we can take the infinite-volume limit only when it has been eliminated. We do not make the finite volume explicit here since it plays no role in the sequel. One may convince oneself that $L^{\Lambda, \Lambda_0}(\phi)$ is equal to

$$L^{\Lambda, \Lambda_0}(\phi) = -\ln Z^{\Lambda, \Lambda_0}((\hat{C}^{\Lambda, \Lambda_0})^{-1}\phi) + 1/2 \langle \phi, (\hat{C}^{\Lambda, \Lambda_0})^{-1}\phi \rangle. \quad (9)$$

Here $Z^{\Lambda, \Lambda_0}(j)$ is the (standard notation for the) generating functional of the Green functions of the (regularized) theory. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in $L_2(\mathbb{R}^4, d^4x)$ so that the second term contains the zero-loop two-point function. Thus $L^{\Lambda, \Lambda_0}(\phi)$ generates the CAG, apart from the order zero contribution given by the inverted free propagator. The functional $L^{\Lambda_0, \Lambda_0}(\phi) = L^0(\phi)$ is the bare action including counterterms, to be calculated from the renormalization conditions. On adding the zero-loop two-point function and including the ϕ^6 -insertion it takes the form (see (1), (2))

$$L^0 = \int \left(\frac{Z}{2} (\partial_\mu \phi)^2(x) + \frac{Z}{2} (m^2 + \delta m^2) \phi^2(x) + \frac{g_0}{4!} Z^2 \phi^4(x) + \frac{Z^3}{\Lambda_0^2} \frac{g_6}{6!} \phi^6(x) \right) d^4x. \quad (10)$$

⁵ Here the ‘fields’ $\phi(x)$ are assumed to be very well behaved, e.g. to live in the Schwartz space $S(\mathbb{R}^4)$. They will drop out from the expressions for the correlation functions in any case.

Here Z , δm^2 and g_0 are formal power series in \hbar . The Wilson FE is a differential equation for the functional L^{Λ, Λ_0} , obtained from (8) on differentiating w.r.t. Λ :

$$\partial_\Lambda (L^{\Lambda, \Lambda_0} + I^{\Lambda, \Lambda_0}) = \frac{\hbar}{2} \left\langle \frac{\delta}{\delta \phi}, (\partial_\Lambda \hat{C}^{\Lambda, \Lambda_0}) \frac{\delta}{\delta \phi} \right\rangle L^{\Lambda, \Lambda_0} - \frac{1}{2} \left\langle \frac{\delta}{\delta \phi} L^{\Lambda, \Lambda_0}, (\partial_\Lambda \hat{C}^{\Lambda, \Lambda_0}) \frac{\delta}{\delta \phi} L^{\Lambda, \Lambda_0} \right\rangle. \tag{11}$$

Changing to momentum space and expanding in a formal power series w.r.t. \hbar we write (with slight abuse of notation)

$$L^{\Lambda, \Lambda_0}(\varphi) = \sum_{l=0}^{\infty} \hbar^l L_l^{\Lambda, \Lambda_0}(\varphi). \tag{12}$$

From $L_l^{\Lambda, \Lambda_0}(\varphi)$ we then obtain the CAG of loop order l in momentum space as⁶

$$(2\pi)^{4(n-1)} \delta_{\varphi(p_1)} \cdots \delta_{\varphi(p_n)} L_l^{\Lambda, \Lambda_0} |_{\varphi \equiv 0} = \delta^{(4)}(p_1 + \cdots + p_n) \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1}) \tag{13}$$

where we have written $\delta_{\varphi(p)} = \delta/\delta\varphi(p)$. Note again that our definition of the $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$ is such that $\mathcal{L}_{0,2}^{\Lambda, \Lambda_0}$ vanishes. This is important for the set-up of the inductive scheme, through which perturbative renormalizability will be established. The FE (11) rewritten in terms of the CAG (13) takes the following form:

$$\begin{aligned} \partial_\Lambda \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1}) &= \frac{1}{2} \int_k (\partial_\Lambda C^{\Lambda, \Lambda_0}(k)) \partial^w \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0}(k, -k, p_1, \dots, p_{n-1}) \\ &\quad - \sum_{\substack{l_1+l_2=l, w_1+w_2+w_3=w \\ n_1+n_2=n}} \frac{1}{2} [\partial^{w_1} \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n_1}) (\partial^{w_3} \partial_\Lambda C^{\Lambda, \Lambda_0}(p')) \\ &\quad \times \partial^{w_2} \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(p_{n_1+1}, \dots, p_n)]_{\text{ssym}} \end{aligned} \tag{14}$$

where

$$p' = -p_1 - \cdots - p_{n_1} = p_{n_1+1} + \cdots + p_n.$$

Here we have written (14) directly in a form where also momentum derivatives of the CAG (13) are performed, and we used the shorthand notation

$$\partial^w := \prod_{i=1}^{n-1} \prod_{\mu=0}^3 \left(\frac{\partial}{\partial p_{i,\mu}} \right)^{w_{i,\mu}} \quad \text{with} \quad w = (w_{1,0}, \dots, w_{n-1,3}) \quad |w| = \sum w_{i,\mu} \in \mathbb{N}_0. \tag{15}$$

The symbol *ssym* (as defined in [KMR]) means summation over those permutations of the momenta p_1, \dots, p_n which do not leave invariant the subsets $\{p_1, \dots, p_{n_1}\}$ and $\{p_{n_1+1}, \dots, p_n\}$. Note that the CAGs are symmetric in their momentum arguments by definition. A simple inductive proof of the renormalizability of ϕ_4^4 theory has been given several times in the literature [KKS, KeKo1, Kop], and we will not repeat it in detail. The line of reasoning can be resumed as follows. The induction hypotheses to be proven are:

(A) *Boundedness.*

$$|\partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(\vec{p})| \leq (\Lambda + m)^{4-n-|w|} \mathcal{P} \left(\log \frac{\Lambda + m}{m} \right) \mathcal{P} \left(\frac{|\vec{p}|}{\Lambda + m} \right). \tag{16}$$

(B) *Convergence.*

$$|\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(\vec{p})| \leq \frac{1}{\Lambda_0^3} \mathcal{P} \left(\log \frac{\Lambda_0}{m} \right) (\Lambda + m)^{6-n-|w|} \mathcal{P} \left(\frac{|\vec{p}|}{\Lambda + m} \right). \tag{17}$$

⁶ The normalization of the $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$ is defined differently from earlier [KeKo1, KKS, Kop].

Here and in the following the \mathcal{P} denote (each time they appear possibly new) polynomials with non-negative coefficients. The coefficients depend on $l, n, |w|, m$, but not on $\vec{p}, \Lambda, \Lambda_0$. We used the shorthand $\vec{p} = (p_1, \dots, p_{n-1})$ and $|\vec{p}| = \sup\{|p_1|, \dots, |p_n|\}$. The statement (17) implies renormalizability: it proves the limits $\lim_{\Lambda \rightarrow 0, \Lambda_0 \rightarrow \infty} \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(\vec{p})$ to exist to all loop orders l . But the statement (16) has to be obtained first to prove (17). To prove (16) we use an inductive scheme that proceeds upwards in $2l + n$, for given $2l + n$ upwards in l , and for given (l, n) downwards in $|w|$, starting from some arbitrary $|w_{\max}| \geq 3$. The important point to note is that the terms on the rhs of the FE (14) always are prior to the ones on the lhs in the inductive order. So the bound (16) may be used as an induction hypothesis on the rhs. Besides we also need a bound on the propagator and its momentum derivatives: it is easy to prove that

$$|\partial^w \partial_\Lambda C^{\Lambda, \Lambda_0}(p)| \leq \Lambda^{-3-|w|} \mathcal{P}(|p|/\Lambda) \exp\left(-\frac{p^2 + m^2}{\Lambda^2}\right). \tag{18}$$

Equipped with this bound and the induction scheme, we may then integrate the FE, where terms with $n + |w| \geq 5$ are integrated down from Λ_0 to Λ , since for those terms we have the boundary conditions following from (10)⁷

$$\partial^w \mathcal{L}_{l,n}^{\Lambda_0, \Lambda_0} \equiv 0 \quad \text{if } n + |w| > 4 \quad \text{and } n \neq 6 \quad \text{and } \mathcal{L}_{l,6}^{\Lambda_0, \Lambda_0} \equiv g_6 \frac{1}{\Lambda_0^2} Z_l^3. \tag{19}$$

The relevant terms (those with $n + |w| \leq 4$) are integrated upwards from 0 to Λ . The boundary conditions for these terms are the renormalization conditions we impose and which fix the counterterms $Z, \delta m^2, g_0$. We may fix for example

$$\mathcal{L}_{l,2}^{0, \Lambda_0}(0) = 0 \quad \partial_{p^2} \mathcal{L}_{l,2}^{0, \Lambda_0}(0) = 0 \quad \mathcal{L}_{l,4}^{0, \Lambda_0}(0) = g \delta_{l,0}. \tag{20}$$

To go away from the renormalization point (here chosen at zero momentum) we may use the Schlömilch or integrated Taylor formula, which takes us back to the irrelevant situation.

The bound (17) holds for the \mathbb{Z}_2 -symmetric theory only; it is sharper than the one from [KKS, KeKo1, Kop] (but generally assumed to be true in the literature). Its proof is based on the same inductive scheme as that for (16). We start from the FE (14) with $|w|$ momentum derivatives applied on it, integrate over Λ and derive w.r.t. Λ_0 . For the terms on the rhs, on which this derivative does not apply, we can use the bound (16). For the terms derived w.r.t. Λ_0 we can use (17), applying our induction scheme. The best bound we can arrive at is essentially saturated by the boundary terms, we find.

We first regard the case of the irrelevant terms with $n + |w| \geq 5$, and here we start looking at the case $n + |w| \geq 6$. We have the equation (in shorthand notation)

$$-\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} = \partial^w (\text{rhs of the FE})|_{\Lambda=\Lambda_0} + \int_{\Lambda}^{\Lambda_0} d\Lambda' \partial_{\Lambda_0} \partial^w (\text{rhs of the FE})(\Lambda'). \tag{21}$$

Now using (16) and (17) to bound the rhs of (21) we verify (17) on performing the integral. In the case $n + |w| = 5$ it suffices to regard zero external momentum (referring again to the Schlömilch formula for deviations from zero, which takes us back to $n + |w| = 6$). In both cases $n = 4, |w| = 1$ and $n = 2, |w| = 3$ the first term of the rhs of (21) is zero due to our boundary conditions, whereas the second vanishes due to Euclidean invariance.

Terms with $n + |w| \leq 4$ have to be analysed at the renormalization point indicated in (20), and they are integrated from 0 to Λ :

$$\partial_{\Lambda_0} \partial^w \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} = \int_0^\Lambda d\Lambda' \partial_{\Lambda_0} \partial^w (\text{rhs of the FE})(\Lambda'). \tag{22}$$

⁷ What we really need in the proof is a bound on $\mathcal{L}_{l,6}^{\Lambda_0, \Lambda_0}$ of the form $|\mathcal{L}_{l,6}^{\Lambda_0, \Lambda_0}| \leq g_6 \frac{1}{\Lambda_0^2} \mathcal{P}(\log \frac{\Lambda_0}{m})$. The bounds on $\mathcal{L}_{l,2}^{\Lambda_0, \Lambda_0}$ are independent of $\mathcal{L}_{l,6}^{\Lambda_0, \Lambda_0}$ (at the same loop order) by the order of the induction scheme. Once these bounds are established, they permit us to bound Z^3 by $\mathcal{P}(\log \frac{\Lambda_0}{m})$, which is then compatible with (19).

Only the second term from (21) appears on the rhs because the renormalization conditions are Λ_0 independent. In this term we may factorize $\Lambda_0^{-3} \mathcal{P}(\log \Lambda_0)$ and verify (17) by induction. As a result of these considerations we obviously obtain the same bounds for the theory with an insertion of ϕ^6 as for the theory with $g_6 = 0$.

To study the theory in the critical domain, and particularly the role of the irrelevant insertions in this domain, we have to analyse the correlation functions in the limit of large correlation length, i.e. in the language of field theory, (the approach to) the massless theory. Thus we regard the propagator

$$C^{\Lambda, \Lambda_0}(p) = \frac{1}{p^2} \left[\exp\left(-\frac{p^2}{\Lambda_0^2}\right) - \exp\left(-\frac{p^2}{\Lambda^2}\right) \right] \quad (\Lambda, p) \neq (0, 0) \quad (23)$$

which is singular for $\sup(\Lambda^2, p^2) \rightarrow 0$. However it has a finite limit for $p^2 \rightarrow 0$, if Λ stays bounded from below so that (18) stays valid for $\Lambda \geq m$. Only the case $\Lambda < m$ has to be reconsidered.

For $\Lambda \rightarrow 0$ in fact the CAG may become singular at certain exceptional momentum configurations, i.e. where subsums of external momenta vanish, but first, for the massless theory to exist at all, certain restrictions on the renormalization conditions have to be observed. More specifically the renormalization points have to be chosen as follows:

$$\mathcal{L}_{l,2}^{0,\Lambda_0}(0) = 0 \quad (\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{l,2}^{0,\Lambda_0})(p=k)|_{\delta_{\mu,\nu}} = 0 \quad \mathcal{L}_{l,4}^{0,\Lambda_0}(k_1, k_2, k_3) = g \delta_{l,0}. \quad (24)$$

This means the mass renormalization has to be performed at zero momentum, whereas the wavefunction renormalization and coupling constant renormalization have to be performed at nonexceptional external momenta, i.e. $k^2 = \mu^2 \neq 0$ and no subsum in k_1, k_2, k_3, k_4 vanishes. Since we have defined the \mathcal{L} to be symmetric functions of their arguments it is natural to make a symmetric⁸ choice, for example, $\vec{k}_i \cdot \vec{k}_j = \frac{\mu^2}{4}(4\delta_{ij} - 1)$ for a fixed nonvanishing momentum scale μ . In (24) $(\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{l,2}^{0,\Lambda_0})(p=k)|_{\delta_{\mu,\nu}}$ refers to the decomposition of the O(4) invariant tensor

$$\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{l,2}^{0,\Lambda_0}(p)|_{p=k} = A(\mu^2)\delta_{\mu,\nu} + B(\mu^2)k_\mu k_\nu$$

and we have defined

$$\partial_{p_\mu} \partial_{p_\nu} \mathcal{L}_{l,2}^{0,\Lambda_0}(p=k)|_{\delta_{\mu,\nu}} = A(\mu^2) \quad (25)$$

so the renormalization condition implies $A(\mu^2) = 0$. Note that $B(\mu^2)$ is irrelevant and need not be fixed by a renormalization condition. Obviously renormalization of the massless theory introduces a new mass scale, which is generally called μ . The problem of exceptional momentum configurations can be studied in full generality and rigour with FEs [KeKo2]: it is possible to define an IR index γ with $2\gamma \in \mathbb{N}$, which measures the exceptionality of the momentum configuration $P = (p_1, \dots, p_n)$. Using the shorthand notations $\mathcal{L} = \mathcal{L}^{0,\infty}$ and $\mathcal{L}^\Lambda = \mathcal{L}^{\Lambda,\infty}$, we may phrase the results from the renormalization proof [KeKo2] for the massless symmetric ϕ_4^4 theory as follows:

- (a) The n -point CAGs with $n > 2$ are smooth functions of the external momenta in the (open) subspace of (arbitrarily) bounded nonexceptional momentum configurations. We have

$$\partial^w \mathcal{L}_{l,n}(p_1, \dots, p_{n-1}) = \lim_{\Lambda \rightarrow 0} \partial^w \mathcal{L}_{l,n}^\Lambda(p_1, \dots, p_{n-1}). \quad (26)$$

- (b) Generally one has

$$|\partial^w \mathcal{L}_{l,n}^\Lambda(p_1, \dots, p_{n-1})| \leq \mu^{4-n-|w|} \left(\frac{\mu}{\Lambda}\right)^{2\gamma(P)+|w|} \mathcal{P}\left(\log \frac{\mu}{\Lambda}\right) \quad 0 < \Lambda \leq \mu. \quad (27)$$

⁸ However, this is not necessary because the solutions of the FE turn out symmetric by construction.

For the two-point function at $\Lambda = 0$ one can also show that it vanishes as $O(p^2 \mathcal{P}(\log \frac{\mu^2}{p^2}))$ near zero momentum.

Since Λ acts as an infrared regulator the bounds (16), (17) still hold for $\Lambda > \mu$, on replacing m by μ . For $\Lambda < \mu$ these bounds also hold for nonexceptional momentum configurations. For exceptional configurations they have to be multiplied by the power of μ/Λ appearing in (27). We do not enter into details of the infrared problem, since the bounds in the region $\Lambda < \mu$ are independent of the ϕ^6 -insertion and therefore the proof from [KeKo2] may be taken over unaltered. As regards the term $\Lambda_0^{-3} \mathcal{P}(\log \frac{\Lambda_0}{\mu})$ appearing in (17), it does not interfere with the exceptional momentum problem and can be factored out in the inductive proof as done before in the massive case.

We now want to establish bounds for the difference between the theories with and without ϕ^6 -insertion. The CAGs of this theory are to be called $\Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$. This means we define (in obvious notation)

$$\Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(p_1, \dots, p_{n-1}) = \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(g_6; p_1, \dots, p_{n-1}) - \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0; p_1, \dots, p_{n-1}). \tag{28}$$

Here it is understood that the $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(g_6)$ and the $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0)$ obey the *same* renormalization conditions, which means that all the relevant $\Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$ are imposed to vanish at the renormalization point. We may obtain the FEs for the $\Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}$ by taking the difference between those for $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(g_6)$ and $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0)$. We only give it in shortened form without momentum arguments, the explicit form following directly from (14). We obtain

$$\begin{aligned} \partial_\Lambda \partial^w \Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0} &= \frac{1}{2} \int_k (\partial_\Lambda C^{\Lambda, \Lambda_0}(k)) \partial^w \Delta \mathcal{L}_{l-1, n+2}^{\Lambda, \Lambda_0}(k, -k, \dots) \\ &\quad - \sum_{\substack{l_1+l_2=l, w_1+w_2+w_3=w \\ n_1+n_2=n}} \frac{1}{2} [\partial^{w_1} \Delta \mathcal{L}_{l_1, n_1+1}^{\Lambda, \Lambda_0} (\partial^{w_3} \partial_\Lambda C^{\Lambda, \Lambda_0}) \partial^{w_2} (\mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(g_6) \\ &\quad + \mathcal{L}_{l_2, n_2+1}^{\Lambda, \Lambda_0}(0))]_{\text{ssym}}. \end{aligned} \tag{29}$$

With this system of equations we can inductively prove the following bounds for the massless theory. For nonexceptional momentum configurations one finds

$$|\partial^w \Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(\vec{p})| \leq \begin{cases} \frac{\mathcal{P}(\log \frac{\Lambda_0}{\mu})}{\Lambda_0^2} \mu^{6-n-|w|} \mathcal{P}\left(\frac{|\vec{p}|}{\mu}\right) & \text{for } 0 \leq \Lambda \leq \mu \\ \frac{\mathcal{P}(\log \frac{\Lambda_0}{\mu})}{\Lambda_0^2} \Lambda^{6-n-|w|} \mathcal{P}\left(\frac{|\vec{p}|}{\Lambda}\right) & \text{for } \mu \leq \Lambda \leq \Lambda_0 \end{cases} \tag{30}$$

whereas for general momentum configurations one obtains

$$|\partial^w \Delta \mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(\vec{p})| \leq \begin{cases} \frac{\mathcal{P}(\log \frac{\Lambda_0}{\mu})}{\Lambda_0^2} \mathcal{P}(\log \mu/\Lambda) \mu^{6-n-|w|} \left(\frac{\mu}{\Lambda}\right)^{2\gamma+|w|} \mathcal{P}\left(\frac{|\vec{p}|}{\mu}\right) & \text{for } 0 \leq \Lambda \leq \mu \\ \frac{\mathcal{P}(\log \frac{\Lambda_0}{\mu})}{\Lambda_0^2} \Lambda^{6-n-|w|} \mathcal{P}\left(\frac{|\vec{p}|}{\Lambda}\right) & \text{for } \mu \leq \Lambda \leq \Lambda_0. \end{cases} \tag{31}$$

We do not give a proof of these bounds, since they are obtained using the same inductive scheme as before, applying also the bounds for $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(g_6)$ and $\mathcal{L}_{l,n}^{\Lambda, \Lambda_0}(0)$ obtained previously. The improvement factor $\mathcal{P}(\log \frac{\Lambda_0}{\mu})/\Lambda_0^2$ is respected in particular by the new boundary conditions: all renormalization conditions vanish, and the only nonvanishing boundary term, i.e. the term $\sim g_6 Z^3/\Lambda_0^2$ for the six-point function, satisfies (30) and (31). Still we would like to point out

that rigorous bounds such as (17), (30) and (31) are hard (if not impossible) to obtain by other methods. We will use them in the next section to obtain equivalent bounds on the corrections to scaling due to irrelevant terms.

3. Renormalization group equations and critical behaviour

We will use the previous results to analyse the modification of critical behaviour by irrelevant terms without composite operator formalism. The advantages of this procedure have been mentioned before. In this last section we will change to the standard notation in the sense that now $\mathcal{L}_2^{0,\Lambda_0}$ denotes the two-point function *including* the zero-loop contribution. Our CAG n -point functions $\mathcal{L}_n^{0,\Lambda_0}$ are defined in terms of the field variable ϕ , which is the renormalized field in standard language. Relating them to the bare functions expressed in terms of the bare field ϕ_B , which is related to ϕ through the relation

$$\phi_B = Z^{1/2} \phi \tag{32}$$

we obtain

$$\mathcal{L}_n^b(p_i, g_0, g_6, \Lambda_0) = Z^{n/2} \left(g, g_6, \frac{\Lambda_0}{\mu} \right) \mathcal{L}_n^{0,\Lambda_0}(p_i, g, g_6, \mu). \tag{33}$$

The sign in the exponent of Z is related to the fact that the functions $\mathcal{L}^{0,\Lambda_0}$ are the connected *free* propagator amputated functions. This sign changes if we use the full propagator amputated functions instead, which is of course possible, but less natural in the FE framework. Taking a derivative of (33) w.r.t. $\ln \mu$ at fixed bare parameters we obtain the (standard) renormalization group equation for the renormalized theory

$$\left[\frac{\partial}{\partial \ln \mu} + \beta \left(g, g_6; \frac{\mu}{\Lambda_0} \right) \frac{\partial}{\partial g} + \frac{1}{2} n \gamma \left(g, g_6; \frac{\mu}{\Lambda_0} \right) \right] \mathcal{L}_n^{0,\Lambda_0}(p_i, g, g_6, \mu) = 0. \tag{34}$$

We have introduced the β and γ functions for the renormalized theory

$$\beta \left(g, g_6; \frac{\mu}{\Lambda_0} \right) = \left. \frac{\partial g}{\partial \ln \mu} \right|_{g_0, g_6, \Lambda_0} \quad \gamma \left(g, g_6; \frac{\mu}{\Lambda_0} \right) = \left. \frac{\partial \ln Z}{\partial \ln \mu} \right|_{g_0, g_6, \Lambda_0}. \tag{35}$$

Since we want to use this equation for large but nevertheless finite Λ_0 the functions $\beta(g, g_6; \frac{\mu}{\Lambda_0})$ and $\gamma(g, g_6; \frac{\mu}{\Lambda_0})$ depend also on Λ_0 . Due to (17) the Λ_0 -dependent terms are bounded by $O((\frac{\mu}{\Lambda_0})^{-2} \mathcal{P}(\log \frac{\Lambda_0}{\mu}))$, since $\beta(g, g_6; \frac{\mu}{\Lambda_0})$ and $\gamma(g, g_6; \frac{\mu}{\Lambda_0})$ may be expressed in terms of $\mathcal{L}^{0,\Lambda_0}$ using (34) for fixed values of n : by dimensional analysis we transform the derivative w.r.t. μ into a derivative w.r.t. p and Λ_0 and obtain from the equations for $n = 4$ and for $n = 2$

$$\begin{aligned} \beta \left(g, g_6; \frac{\mu}{\Lambda_0} \right) &= \sum_{i=1}^3 p_{i,v} \frac{\partial}{\partial p_{i,v}} \mathcal{L}_4^{0,\Lambda_0}(g, g_6)|_{r.p.} - 4g p^2 \frac{\partial^2}{\partial (p^2)^2} \mathcal{L}_2^{0,\Lambda_0}(g, g_6)|_{r.p.} \\ &+ O \left(\left(\frac{\mu}{\Lambda_0} \right)^2 \mathcal{P} \left(\log \frac{\Lambda_0}{\mu} \right) \right) \quad \gamma \left(g, g_6; \frac{\mu}{\Lambda_0} \right) = 2 p^2 \frac{\partial^2}{\partial (p^2)^2} \mathcal{L}_2^{0,\Lambda_0}|_{r.p.} \\ &+ O \left(\left(\frac{\mu}{\Lambda_0} \right)^2 \mathcal{P} \left(\log \frac{\Lambda_0}{\mu} \right) \right). \end{aligned} \tag{36}$$

The functions are to be taken at the renormalization points (see (24)). The contributions $\sim O(\Lambda_0^{-2} \mathcal{P} \log \Lambda_0)$ arise when transforming the μ -derivative into one on Λ_0 on using the bound (17). So to be precise we rewrite (34) as

$$\left[\frac{\partial}{\partial \ln \mu} + \beta(g, g_6) \frac{\partial}{\partial g} + \frac{1}{2} n \gamma(g, g_6) \right] \mathcal{L}_n^{0,\Lambda_0}(p_i, g, g_6, \mu) = O \left(\left(\frac{\mu}{\Lambda_0} \right)^2 \mathcal{P} \left(\log \frac{\Lambda_0}{\mu} \right) \right) \tag{37}$$

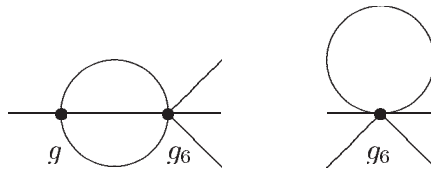


Figure 1. Contributions $\sim g_6$ to the relation between g and g_0 up to two loops.

where the whole dependence on Λ_0 has been regrouped on the rhs (with the definitions $\beta(g, g_6) = \beta(g, g_6; 0)$, $\gamma(g, g_6) = \gamma(g, g_6; 0)$). When setting $g_6 = 0$ we obtain the corresponding equation with functions $\beta(g, 0; \frac{\mu}{\Lambda_0})$ and $\gamma(g, 0; \frac{\mu}{\Lambda_0})$ obeying the equations analogous to (36) for $g_6 = 0$. From this it follows on using (30) that

$$\Delta\beta\left(g, g_6; \frac{\mu}{\Lambda_0}\right) := \beta\left(g, g_6; \frac{\mu}{\Lambda_0}\right) - \beta\left(g, 0; \frac{\mu}{\Lambda_0}\right) = \mathcal{O}\left(\left(\frac{\mu}{\Lambda_0}\right)^2 \mathcal{P}\left(\log\frac{\Lambda_0}{\mu}\right)\right) \quad (38)$$

and similarly for γ . This bound can of course be verified in lowest orders by direct calculation of the respective β -functions. When the ϕ^6 -term is added, the two diagrams given in figure 1 contribute to the relation between g and g_0 and thus to $\beta(g, g_6; \frac{\mu}{\Lambda_0})$ up to two loops⁹. Since the second diagram is μ independent, only the first contributes to the β -function. The value of the diagram is

$$g g_6 \frac{2}{16\pi^4} \ln \frac{4}{3} + \mathcal{O}\left(\left(\frac{\mu}{\Lambda_0}\right)^2 \log\left(\frac{\Lambda_0}{\mu}\right)\right)$$

so that after derivation w.r.t. $\ln \mu$ its contribution is of the order given in (38).

We refer to the textbooks [ZJ, IZ] for the method of solution of (34), which permits us to compare $\mathcal{L}_n^{0, \Lambda_0}(p_i, g, g_6, \mu)$ with $\mathcal{L}_n^{0, \Lambda_0}(\frac{p_i}{s}, g(s), g_6, \mu)$, the critical region corresponding to $s \rightarrow \infty$. Here the running coupling at scale μ/s is defined through

$$\frac{dg(s)}{d \ln s} = -\beta(g(s)) \quad g(1) = g. \quad (39)$$

From (34), together with dimensional analysis, one obtains

$$\mathcal{L}_n^{0, \Lambda_0}(p_i, g, g_6, \mu) = s^{-4+n} \exp\left(\frac{1}{2}n \int_g^{g(s)} \frac{\gamma(g', g_6; \mu/\Lambda_0)}{\beta(g', g_6; \mu/\Lambda_0)} dg'\right) \mathcal{L}_n^{0, s\Lambda_0}(sp_i, g(s), g_6, \mu) \quad (40)$$

or on replacing p_i by p_i/s

$$\mathcal{L}_n^{0, \Lambda_0}\left(\frac{p_i}{s}, g, g_6, \mu\right) = s^{-4+n} \exp\left(\frac{1}{2}n \int_g^{g(s)} \frac{\gamma}{\beta} dg'\right) \mathcal{L}_n^{0, s\Lambda_0}(p_i, g(s), g_6, \mu). \quad (41)$$

For $s \gg 1$ the coupling will approach its fixed-point value g^* for which by definition $\beta(g^*) = 0$. In the perturbative region we have $g^* = 0$ in $d = 4$, whereas in $d < 4$ one finds $g^* = \mathcal{O}(\varepsilon)$ with $\varepsilon = 4 - d$. If g is in the vicinity of the fixed point the integral $\int_g^{g(s)} \frac{\gamma(g')}{\beta(g')} dg'$ is approximated by its value at g^*

$$-\int_g^{g(s)} \frac{\gamma(g')}{\beta(g')} dg' = \int_0^{\ln s} \gamma(g(s')) d \ln s' \sim \gamma(g^*) \ln s. \quad (42)$$

The neglected terms give subdominant contributions for $s \rightarrow \infty$; they are analysed in [BGZ]. From this we then find for the dominating behaviour

$$\mathcal{L}_n^{0, \Lambda_0}\left(\frac{p_i}{s}, g, g_6, \mu\right) \sim s^{-4+n(1-\frac{\gamma(g^*)}{2})} \mathcal{L}_n^{0, \infty}(p_i, g^*, g_6, \mu) \quad (43)$$

⁹ We have not included those diagrams which are exactly cancelled by diagrams carrying an insertion of a counterterm.

which shows that the fixed-point value $\gamma(g^*)$ is to be identified with the critical exponent η .

The renormalization group equation for the difference functions (28) can be obtained from (34). We write it in the form

$$\begin{aligned} & \left[\frac{\partial}{\partial \ln \mu} + \beta \left(g, 0; \frac{\mu}{\Lambda_0} \right) \frac{\partial}{\partial g} + \frac{n}{2} \gamma \left(g, 0; \frac{\mu}{\Lambda_0} \right) \right] \Delta \mathcal{L}_n^{0, \Lambda_0}(p_i, g, g_6, \mu) \\ &= - \left[\Delta \beta \left(g, g_6; \frac{\mu}{\Lambda_0} \right) \frac{\partial}{\partial g} + \frac{n}{2} \Delta \gamma \left(g, g_6; \frac{\mu}{\Lambda_0} \right) \right] \mathcal{L}_n^{0, \Lambda_0}(p_i, g, g_6, \mu). \end{aligned} \tag{44}$$

For the inhomogeneous equation we make the ansatz

$$\Delta \mathcal{L}_n^{0, \Lambda_0}(p_i, g, g_6, \mu) = U_n^{0, \Lambda_0}(p_i, g, g_6, \mu) \mathcal{L}_n^{0, \Lambda_0}(p_i, g, 0, \mu). \tag{45}$$

From this we obtain the following differential equation for U^{0, Λ_0} :

$$\begin{aligned} & \left(\frac{\partial}{\partial \ln \mu} + \beta \left(g, 0; \frac{\mu}{\Lambda_0} \right) \frac{\partial}{\partial g} \right) U^{0, \Lambda_0} \\ &= - \left[\Delta \beta \left(g, g_6; \frac{\mu}{\Lambda_0} \right) \frac{\partial}{\partial g} \ln \mathcal{L}_n^{0, \Lambda_0}(p_i, g, 0, \mu) + \frac{n}{2} \Delta \gamma \left(g, g_6; \frac{\mu}{\Lambda_0} \right) \right] \\ & \quad + \mathcal{O} \left(\left(\frac{\mu}{\Lambda_0} \right)^{-4} \mathcal{P} \left(\log \frac{\Lambda_0}{\mu} \right) \right). \end{aligned} \tag{46}$$

In the following we will neglect the last term, which gives even smaller corrections; for the first two terms on the rhs of this equation we write $V_n(p_i, \mu, g; g_6, \Lambda_0)$. Its solution is then obtained as a sum of the general solution of the corresponding homogeneous equation—which in turn is obtained as previously for the case $\gamma = 0$ —plus a special solution of the inhomogeneous equation, which can be written as the integral over $V_n(p_i, \mu, g; g_6, \Lambda_0)$. As a final result we obtain the following renormalization group relation for $U_n^{0, \Lambda_0}(p_i, g, g_6, \mu)$:

$$U_n^{0, \Lambda_0}(p_i, g, g_6, \mu) = U_n^{0, \Lambda_0}(p_i, g(s), g_6, \mu/s) + \int_{-\ln s}^0 V_n(p_i, \mu e^t, g(e^{-t}); g_6, \Lambda_0) dt. \tag{47}$$

By dimensional analysis we obtain

$$U_n^{0, \Lambda_0}(p_i, g, g_6, \mu) = U_n^{0, s\Lambda_0}(s p_i, g(s), g_6, \mu) + \int_{-\ln s}^0 V_n(s p_i, s \mu e^t, g(e^{-t}); g_6, s \Lambda_0) dt \tag{48}$$

since the canonical dimension of U_n is zero. Multiplying by $\mathcal{L}_n^{0, \Lambda_0}(p_i, g, 0, \mu)$, using (41) and passing to momenta p_i/s we thus obtain

$$\begin{aligned} \Delta \mathcal{L}_n^{0, \Lambda_0} \left(\frac{p_i}{s}, g, g_6, \mu \right) &= s^{-4+n} \exp \left(\frac{1}{2} n \int_g^{g(s)} \frac{\gamma(g', 0; \mu/\Lambda_0)}{\beta(g', 0; \mu/\Lambda_0)} dg' \right) \left[\Delta \mathcal{L}_n^{0, s\Lambda_0}(p_i, g(s), g_6, \mu) \right. \\ & \quad \left. + \mathcal{L}_n^{0, s\Lambda_0}(p_i, g(s), 0, \mu) \int_{-\ln s}^0 V_n(p_i, s \mu e^t, g(e^{-t}); g_6, s \Lambda_0) dt \right]. \end{aligned} \tag{49}$$

The second term can be bounded using (38) (together with (16))¹⁰; to the first term we can apply (30) to obtain the following bound on $\Delta \mathcal{L}_n^{0, \Lambda_0}(\frac{p_i}{s}, g, g_6, \mu)$:

$$\begin{aligned} \left| \Delta \mathcal{L}_n^{0, \Lambda_0} \left(\frac{p_i}{s}, g, g_6, \mu \right) \right| &\leq s^{-4+n} \exp \left(\frac{1}{2} n \int_g^{g(s)} \frac{\gamma}{\beta} dg' \right) \left[\mathcal{O} \left(\left(\frac{\mu}{s \Lambda_0} \right)^2 \mathcal{P} \left(\log \frac{s \Lambda_0}{\mu} \right) \right) \right. \\ & \quad \left. + |\mathcal{L}_n^{0, s\Lambda_0}(p_i, g(s), 0, \mu)| \mathcal{O} \left(\left(\frac{\mu}{\Lambda_0} \right)^2 \mathcal{P} \left(\log \frac{\Lambda_0}{\mu} \right) \right) \right]. \end{aligned} \tag{50}$$

¹⁰ It is useful to cut the integration interval into subintervals of length $\ln 2$ and sum over the bounds for the integrand in the subintervals to avoid a factor of $\ln s$ in the bound for this term.

This bound is dominated for s large by the second term so that we obtain

$$\left| \Delta \mathcal{L}_n^{0, \Lambda_0} \left(\frac{p_i}{s}, g, g_6, \mu \right) \right| \leq s^{-4+n} \exp \left(\frac{1}{2} n \int_g^{g(s)} \frac{\gamma}{\beta} dg' \right) |\mathcal{L}_n^{0, s\Lambda_0}(p_i, g(s), 0, \mu)| \\ \times \mathcal{O} \left(\left(\frac{\mu}{\Lambda_0} \right)^2 \mathcal{P} \left(\log \frac{\Lambda_0}{\mu} \right) \right). \quad (51)$$

The analysis of the prefactor is the same as for $\mathcal{L}_n^{0, \Lambda_0}(\frac{p_i}{s}, g, 0, \mu)$. Therefore, close to the critical region, the corrections of the long-distance behaviour due to the irrelevant term are of the relative order $\mathcal{O}((\frac{\mu}{\Lambda_0})^2)$ up to logarithms. For this term to be negligible we need of course $\mu \ll \Lambda_0$, that is to say, the renormalized parameters are close to the critical ones, which is a natural parametrization in the critical region. We emphasize that the corrections to scaling stem from the analysis of the terms vanishing for $\Lambda_0 \rightarrow \infty$, which are often neglected altogether in the literature. In the composite operator analysis one finds instead corrections $\sim s^{-2} \mathcal{P}(\log s)$, which would be smaller for $s > \frac{\Lambda_0}{\mu}$. However the terms $\sim (\frac{\mu}{\Lambda_0})^2$ are always present, though often neglected, so that the corresponding results only hold up to $s \sim \frac{\Lambda_0}{\mu}$. For larger s one has to readapt the renormalization conditions at $\mu' \ll \mu$. In terms of the bare theory the readaptation consists in adding new counterterms $\sim \phi^2$ and $\sim \phi^4$. This is well known from the treatment in the composite operator formalism, where such terms are introduced due to operator mixing.

In conclusion we thus realize that in our approach the corrections to scaling due to irrelevant terms are suppressed by $\mathcal{O}((\frac{\mu}{\Lambda_0})^2)$ to any order in the number of insertions. These irrelevant terms are introduced directly in the bare action, keeping the renormalization conditions fixed. In composite operator theory, which is completely bypassed here, the coefficient of the ϕ^6 -term in the bare action is not suppressed by $(\frac{1}{\Lambda_0})^2$, correspondingly one does not obtain such a suppression in the corrections to scaling. Instead, on subtracting insertions of lower dimension, to be calculated from the relations for operator mixing, one obtains a suppression factor $\sim s^{-2}$, which is larger for $s < \frac{\Lambda_0}{\mu}$, becomes of similar size for $s \sim \frac{\Lambda_0}{\mu}$ and is unreliable beyond.

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